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(m, p) -hyperexpansive mappings on metric spaces

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Abstract. In the present paper, we define the concept of (m, p) -hyperexpansive mappings in metric space, which are the extension of (m, p) -isometric mappings recently introduced in [13]. We give a first approach of the general theory of these maps.

Keywords: metric space, (m, p) -isometric, expansive maps, hyperexpansive maps.

MSC 2010 classification: primary 54E40, secondary 47B99.

1 Introduction and notations

The introduction of the concept of m -isometric transformation in Hilbert spaces by Agler and Stankus yielded a flow of papers generalizing this concept both in Hilbert and Banach spaces, for example (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 22]).

An operator T acting on a Hilbert space \mathcal{H} is called m -isometric for some integer $m \geq 1$ if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0 \quad (1.1)$$

where $\binom{m}{k}$ be the binomial coefficient. A simple manipulation proves that (1.1) is equivalent to

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \quad \text{for all } x \in \mathcal{H} \quad (1.2)$$

Evidently, an isometric operator (i.e., a 1-isometric operator) is an m -isometric for all integers $m \geq 1$. Indeed the class of m -isometric operators is a generalization of the class of isometric operators and a detailed study of this class and in particular 2-isometric operators on a Hilbert space has been the object of some

intensive study, especially by J. Agler and M. Stankus in [2], [3] and [4], but also by S.M. Patel [23]. B.P. Duggal [15, 16] studied when the tensor product of operators is an m -isometry.

A generalization of m -isometries to operators on general Banach spaces has been presented by several authors in the last years. Botelho [14] and Sid Ahmed [21] discussed operators defined via (1.2) on (complex) Banach spaces. Bayart introduced in [8] the notion of (m, p) -isometries on general (real or complex) Banach spaces. An operator T on a Banach space X into itself is called an (m, p) -isometry if there exists an integer $m \geq 1$ and a $p \in [1, \infty)$, with

$$\forall x \in X, \quad \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|T^{m-k}x\|^p = 0 \quad (1.3)$$

It is easy to see that, if $X = \mathcal{H}$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of m -isometries. In [19] the authors took off the restriction $p \geq 1$ and defined (m, p) -isometries for all $p > 0$. They studied when an (m, p) -isometry is an (μ, q) -isometry for some pair (μ, q) . In particular, for any positive real number p they gave an example of an operator T that is a $(2, p)$ -isometry, but is not a $(2, q)$ -isometry for any q different from p . In [9, 10] it is proven that the powers of an m -isometry are m -isometries and some products of m -isometries are again m -isometries.

The authors, O.A.M. Sid Ahmed and A. Saddi introduced the concept of (A, m) -isometric operators. They gave several generalizations of well known facts on m -isometric operators according to semi-Hilbertian space structures. We refer the reader to [22] for more details about (A, m) -isometric operators. Recently, B.P. Duggal has introduced the concept of an $A(m, p)$ -isometry of a Banach space, following a definition of Bayart in the Banach space.

Definition 1.1. ([17]) Let T and $A \in \mathcal{B}(X)$ (the set of bounded linear operators from X into itself), m is a positive integer and $p > 0$ a real number. We say that T is an $A(m, p)$ -isometry if, for every $x \in X$

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|AT^{m-k}x\|^p = 0. \quad (1.4)$$

For any $T \in \mathcal{B}(\mathcal{H})$ we let

$$\theta_m(T) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*k} T^k. \quad (1.5)$$

The Concept of completely hyperexpansive operators on Hilbert space has attracted much attention of various authors. In [1], J. Agler characterized subnormality with the positivity of $\theta_m(T)$ in (1.5) and also extended his inequalities to

the concept of m -isometry (cf. [2–4]). On the other hand, A. Athavale considered completely hyperexpansive operators in [5]. In further studies, many authors have studied k -hyperexpansive (cf. [7, 18]). The concept of (A, m) -expansive operators on Hilbert space was introduced in [20].

Definition 1.2. ([18]) An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

- (i) m -isometry ($m \geq 1$) if $\theta_m(T) = 0$.
- (ii) m -expansive ($m \geq 1$) if $\theta_m(T) \leq 0$.
- (iii) m -hyperexpansive ($m \geq 1$), if $\theta_k(T) \leq 0$ for $k = 1, 2, \dots, m$.
- (iv) Completely hyperexpansive if $\theta_m(T) \leq 0$ for all m .

We refer the reader to [6, 7, 18] for recent articles concerning this subject.

In [8] the author defined $\beta_k^{(p)}(T, \cdot) : X \rightarrow \mathbb{R} : x \mapsto \beta_k^{(p)}(T, x)$ by

$$\beta_k^{(p)}(T, x) = \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} \|T^j x\|^p, \quad \forall x \in X \quad (1.6)$$

For $k, n \in \mathbb{N}$ denote the (descending Pochhammer) symbol by $n^{(k)}$, i.e.

$$n^{(k)} = \begin{cases} 0, & \text{if } n = 0 \\ 0 & \text{if } n > 0 \text{ and } k > n \\ \binom{n}{k} k! & \text{if } n > 0 \text{ and } k \leq n. \end{cases}$$

Then for $n > 0$, $k > 0$ and $k \leq n$ we have

$$n^{(k)} = n(n-1)\dots(n-k+1).$$

It was proved in [8, Proposition 2.1] that

$$\|T^n x\|^p = \sum_{0 \leq k \leq m-1} n^{(k)} \beta_k^{(p)}(T, x) \quad (1.7)$$

for all integers $n \geq 0$ and $x \in X$. In particular,

$$\beta_{m-1}^{(p)}(T, x) = \lim_{n \rightarrow \infty} \frac{\|T^n x\|^p}{\binom{n}{m-1} (m-1)!} \geq 0$$

with equality if and only if T is $(m-1; p)$ -isometric.

In recent work T. Bermúdez, A. Martínón and V. Müller introduced the concept of (m, p) -isometric maps on metric spaces (see [13]).

Let \mathbb{N} be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.3. ([13]) Let E be a metric space. A map $T : E \longrightarrow E$ is called an (m, p) -isometry, ($m \geq 1$ integer and $p > 0$) if, for all $x, y \in E$

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(T^{m-k}x, T^{m-k}y)^p = 0. \quad (1.8)$$

For $m \geq 2$, T is a strict (m, p) -isometry if it is an (m, p) -isometry, but is not an $(m-1, p)$ -isometry.

For any $p > 0$; $(1, p)$ -isometry coincide with isometry, that is $d(Tx, Ty) = d(x, y)$ for all $x, y \in E$. Every isometry is an (m, p) -isometry for all $m \geq 1$ and $p > 0$. Many results known in the Banach space setting are established in [13] for metric spaces. For example, an (m, p) -isometry is an $(m+1, p)$ -isometry and any power of (m, p) -isometry is again an (m, p) -isometry.

Let $T : E \longrightarrow E$ is an (m, p) -isometry. In [13] the authors defined $f_T(h, p, x, y)$ for $h \in \mathbb{N}$, a positive real number p and $x, y \in E$ by :

$$f_T(h, p, x, y) = \sum_{0 \leq k \leq h} (-1)^{h-k} \binom{h}{k} d(T^k x, T^k y)^p. \quad (1.9)$$

We have from (1.9) that

$$d(T^n x, T^n y)^p = \sum_{0 \leq k \leq m-1} \binom{n}{k} f_T(k, p; x, y). \quad (1.10)$$

for all $n \geq 0$ and $x, y \in X$ (see [13]).

Definition 1.4. ([5]) A real-valued function Ψ on \mathbb{N}_0 is said to be

(1) completely monotone if $\Psi \geq 0$ and $\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \Psi(n+k) \geq 0, \forall n \geq 0$ and $m \geq 1$.

(2) completely alternating if $\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \Psi(n+k) \leq 0, \forall n \geq 0$ and $m \geq 1$.

The content of this paper is as follows. In Section one we set up notation and terminology. Furthermore, we collect some facts about (m, p) -isometries. In Section two, we introduce and study the concept of (m, p) -expansive and hyperexpansive mappings on a metric space and we investigate various structural properties of this classes of mappings. We prove that $(2, p)$ -hyperexpansive mappings which are (m, p) -expansive must be $(m-1, p)$ -expansive for $m \geq 2$. Recall that if T is an m -isometry (resp. k -expansive or (A, m) -expansive) operator, then so

are all its power T^n ; for $n \geq 1$ (cf [9, 18, 20]). It turns out that the same assertion remains true for $(2, p)$ -hyperexpansive and completely p -hyperexpansive mapping (Theorem 2.3 and Theorem 2.4). Moreover, we prove that the intersection of the class of completely p -hyperexpansive mapping and the class of (m, p) -isometries for $m \geq 2$ is the class of $(2, p)$ -isometries (Proposition 2.10). The section three of this paper is an attempt to develop some properties of the class of (m, p) -expansive mappings in seminormed spaces parallel to those of m -isometries.

2 (m, p) -Hyperexpansive maps in metric spaces

In this section, let (X, d) be a metric space, $T : X \rightarrow X$ is a map, $m \in \mathbb{N}$ and $p > 0$ is a real number. We define the quantity

$$\Theta_m^{(p)}(d, T; x, y) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(T^k x, T^k y)^p,$$

for all $x, y \in X$ and we give several results on (m, p) expansive and hyperexpansive mappings on a metric space.

In the following definition, $\Theta_m^{(p)}(d, T; x, y) \leq 0$ (resp. $\Theta_m^{(p)}(d, T; x, y) \geq 0$) really means $\Theta_m^{(p)}(d, T; x, y) \leq 0$ for all $x, y \in X$ (resp. $\Theta_m^{(p)}(d, T; x, y) \geq 0$ for all $x, y \in X$).

Definition 2.1. Let $T : X \rightarrow X$ be a map. We say that

- (i) T is (m, p) -expansive if $\Theta_m^{(p)}(d, T; x, y) \leq 0$.
- (ii) T is (m, p) -hyperexpansive if $\Theta_k^{(p)}(d, T; x, y) \leq 0$ for $k = 1, 2, \dots, m$.
- (iii) T is completely p -hyperexpansive if T is (k, p) -expansive for all $k \in \mathbb{N}$.
- (iv) T is (m, p) -contractive if $\Theta_m^{(p)}(d, T; x, y) \geq 0$.
- (v) T is (m, p) -hypercontractive if $\Theta_k^{(p)}(d, T; x, y) \geq 0$ for $k = 1, 2, \dots, m$.
- (vi) T is completely p -hypercontractive if T is (k, p) -contractive for all $k \in \mathbb{N}$.

For any $p > 0$, $(1, p)$ -expansive coincide with expansive; that is, maps T satisfying $d(Tx, Ty) \geq d(x, y)$, for all $x, y \in X$.

For any $p > 0$, $(1, p)$ -contractive coincide with contractive; that is, maps T satisfying $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in X$. (m, p) -isometries maps are special cases of the class of (m, p) -expansive and contractive maps.

We consider the following examples of (m, p) -expansive map and (m, p) -contractive map which are not (m, p) -isometric map.

Example 2.1. Let $X = \mathbb{R}$ be equipped with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \rightarrow X$ by $Tx = 2x$. Clearly $\Theta_m^{(p)}(d, T; x, y) = (1 - 2^p)^m |x - y|^p$. So we can say that T is neither (m, p) -isometric, for all $m \geq 1$ and $p > 0$. However, one can easily verify that T is (m, p) expansive map for positive odd integer m and (m, p) -contractive map for positive even integer m .

Remark 2.1. Every (m, p) -expansive map T is injective. In fact if $Tx = Ty$ then $T^k x = T^k y$ for $k = 1, 2, \dots, m$ and from (i) of Definition 2.1 we obtain $d(x, y) \leq 0$ i.e $x = y$. Hence T is an injective map.

We note that an (m, p) -expansive map is in general not an $(m + 1, p)$ -expansive, as we shown in the following example.

Example 2.2. Consider the usual metric $d(x, y) = |x - y|$ on \mathbb{R} . Let $T : (\mathbb{R}, d) \rightarrow (\mathbb{R}, d)$ defined by $Tx = 1 + 2x$. Then it is easy to see that $d(Tx, Ty) \geq d(x, y)$ and

$$d(T^2 x, T^2 y)^p - 2d(Tx, Ty)^p + d(x, y)^p = (2^p - 1)^2 |x - y|^p \not\leq 0.$$

Clearly T is $(1, p)$ -expansive which is not $(2, p)$ -expansive.

Remark 2.2. We note the following:

- (1) $\Theta_m^{(p)}(d, T; x, y) \leq 0 \iff \Theta_m^{(p)}(d, T; T^n x; T^n y) \leq 0, \forall x, y \in X, \forall n \in \mathbb{N}_0$.
- (2) $\Theta_m^{(p)}(d, T; x, y) \geq 0 \iff \Theta_m^{(p)}(d, T; T^n x; T^n y) \geq 0, \forall x, y \in X, \forall n \in \mathbb{N}_0$.

Remark 2.3. We deduce from ([5], Proposition 1 and Proposition 2) the following characterizations of completely p -hyperexpansive and completely p -hypercontractive maps.

- (1) A map $T : X \rightarrow X$ is completely p -hyperexpansive if and only if for every $x, y \in X$, the map $n \mapsto \Psi_{(T, p, x, y)}(n) = d(T^n x, T^n y)^p$ is completely alternating.
- (2) A map $T : X \rightarrow X$ is completely p -hypercontractive if and only if for every $x, y \in X$, the map $n \mapsto \Psi_{(T, p, x, y)}(n) = d(T^n x, T^n y)^p$ is completely monotone.

In the next proposition we invoke the following relation which plays an important role in the proof of main results.

Proposition 2.1. For a map $T : X \rightarrow X$, $m \in \mathbb{N}$, real number $p > 0$ and $x, y \in X$, we have that

$$\Theta_m^{(p)}(d, T; x, y) = \Theta_{m-1}^{(p)}(d, T; x, y) - \Theta_{m-1}^{(p)}(d, T; Tx, Ty). \quad (2.1)$$

Proof. By the standard formula $\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$ for binomial coefficients we have the equalities

$$\begin{aligned}
 & \Theta_m^{(p)}(d, T; x, y) \\
 = & \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(T^k x, T^k y)^p \\
 = & d(x, y)^p + \sum_{1 \leq k \leq m-1} (-1)^k \binom{m}{k} d(T^k x, T^k y)^p + (-1)^m d(T^m x, T^m y)^p \\
 = & d(x, y)^p + \sum_{1 \leq k \leq m-1} (-1)^k \left(\binom{m-1}{k} + \binom{m-1}{k-1} \right) d(T^k x, T^k y)^p + \\
 & + (-1)^m d(T^m x, T^m y)^p \\
 = & \Theta_{m-1}^{(p)}(d, T; x, y) - \Theta_{m-1}^{(p)}(d, T; Tx, Ty).
 \end{aligned}$$

$\square QED$

Remark 2.4. We note the following equivalences:

- (1) T is (m, p) – expansive $\iff \forall x, y \in X$
- $$\sum_{\substack{0 \leq k \leq m \\ k \text{ even}}} \binom{m}{k} d(T^k x, T^k y)^p \leq \sum_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \binom{m}{k} d(T^k x, T^k y)^p$$
- (2) T is (m, p) – contractive $\iff \forall x, y \in X$
- $$\sum_{\substack{0 \leq k \leq m \\ k \text{ even}}} \binom{m}{k} d(T^k x, T^k y)^p \geq \sum_{\substack{0 \leq k \leq m \\ k \text{ odd}}} \binom{m}{k} d(T^k x, T^k y)^p$$

Lemma 2.1. Let $T : X \rightarrow X$ be an $(2, p)$ -expansive mapping. Then the following properties hold

- (1) $d(Tx, Ty)^p \geq \frac{n-1}{n} d(x, y)^p$, $n \geq 1$, $x, y \in X$.
- (2) $d(Tx, Ty)^p \geq d(x, y)^p$ for all $x, y \in X$.
- (3) $d(T^n x, T^n y)^p + (n-1)d(x, y)^p \leq n.d(Tx, Ty)^p$, $x, y \in X$, $n = 0, 1, 2, \dots$
- (4) $d(Tx, Ty) \leq 2^{\frac{1}{p}} d(x, y) \quad \forall x, y \in \mathcal{R}(T)$ (the range of T).

Proof. Using the fact that T is $(2, p)$ -expansive map, we get

$$d(T^2x, T^2y)^p - d(Tx, Ty)^p \leq d(Tx, Ty)^p - d(x, y)^p.$$

Replacing x by T^kx and y by T^ky leads to

$$d(T^{k+2}x, T^{k+2}y)^p - d(T^{k+1}x, T^{k+1}y)^p \leq d(T^{k+1}x, T^{k+1}y)^p - d(T^kx, T^ky)^p,$$

for $k \geq 0$. Hence

$$\begin{aligned} d(T^n x, T^n y)^p &= \sum_{1 \leq k \leq n} (d(T^k x, T^k y)^p - d(T^{k-1} x, T^{k-1} y)^p) + d(x, y)^p \\ &\leq n(d(Tx, Ty)^p - d(x, y)^p) + d(x, y)^p \\ &\leq nd(Tx, Ty)^p + (1 - n)d(x, y)^p. \end{aligned}$$

Which implies 1. and 3. Letting $n \rightarrow \infty$ in 1. yields 2.

4. The $(2, p)$ -expansivity of T implies that

$$d(T^2x, T^2y)^p \leq 2d(Tx, Ty)^p - d(x, y)^p \leq 2d(Tx, Ty)^p.$$

Thus,

$$d(T^2x, T^2y) \leq 2^{\frac{1}{p}} d(Tx, Ty).$$

\square *QED*

Remark 2.5. We make the following remarks:

- (1) $(2, p)$ -isometric is completely p -hyperexpansive.
- (2) Every $(k + 1, p)$ -hyperexpansive is (k, p) -hyperexpansive for $k = 1, 2, \dots$

Lemma 2.2. Let $T : X \rightarrow X$ be an $(2, p)$ -expansive map, then for all integer $k \geq 2$ and $x, y \in X$, we have

$$d(T^k x, T^k y)^p - d(T^{k-1} x, T^{k-1} y)^p \leq d(Tx, Ty)^p - d(x, y)^p.$$

Proof. We prove the assertion by induction on k . Since T is an $(2, p)$ -expansive the result is true for $k = 2$. Now assume that the result is true for k i.e.; for all $x, y \in X$,

$$d(T^k x, T^k y)^p - d(T^{k-1} x, T^{k-1} y)^p \leq d(Tx, Ty)^p - d(x, y)^p, \quad (2.2)$$

and let us prove it of $k + 1$. From (2.2) we obtain the following inequalities

$$\begin{aligned} d(T^{k+1} x, T^{k+1} y)^p - d(T^k x, T^k y)^p &\leq d(T^2x, T^2y)^p - d(Tx, Ty)^p \\ &\leq d(Tx, Ty)^p - d(x, y)^p. \end{aligned}$$

\square *QED*

Proposition 2.2. Let $T : X \longrightarrow X$ be a $(2, p)$ -expansive map. Then the following statements hold.

- (1) T is $(2, p)$ -hyperexpansive map.
- (2) $d(Tx, Ty)^{2p} \geq d(x, y)^p d(T^2x, T^2y)^p$ for all $x, y \in X$.
- (3) For each n and $x, y \in X$ such that $x \neq y$, the sequence

$$\left(\frac{d(T^{n+1}x, T^{n+1}y)^p}{d(T^nx, T^ny)^p} \right)_{n \geq 0} \quad (2.3)$$

is monotonically decreasing to 1.

Proof. (1) Follows from part (2) of Lemma 2.1.

(2) Since from (1) T is $(2, p)$ -hyperexpansive map, we have that

$$\begin{aligned} d(Tx, Ty)^{2p} &\geq \left(\frac{d(x, y)^p + d(T^2x, T^2y)^p}{2} \right)^2 \\ &\geq \left(d(x, y)^{\frac{p}{2}} d(T^2x, T^2y)^{\frac{p}{2}} \right)^2 \\ &\geq d(x, y)^p d(T^2x, T^2y)^p. \end{aligned}$$

(3) Observe that the $(2, p)$ -expansivity of T implies that

$$d(T^{n+1}x, T^{n+1}y)^p - 2d(T^nx, T^ny)^p + d(T^{n-1}x, T^{n-1}y)^p \leq 0. \quad (2.4)$$

On the other hand, since

$$\left(d(T^{n-1}x, T^{n-1}y)^{\frac{p}{2}} - d(T^{n+1}x, T^{n+1}y)^{\frac{p}{2}} \right)^2 \geq 0$$

it follows that

$$\begin{aligned} &d(T^{n-1}x, T^{n-1}y)^{\frac{p}{2}} d(T^{n+1}x, T^{n+1}y)^{\frac{p}{2}} \\ &\leq \frac{d(T^{n+1}x, T^{n+1}y)^p + d(T^{n-1}x, T^{n-1}y)^p}{2} \\ &\leq d(T^nx, T^ny)^p \quad (\text{by (2.4)}). \end{aligned}$$

Thus,

$$d(T^{n-1}x, T^{n-1}y)^p d(T^{n+1}x, T^{n+1}y)^p \leq d(T^nx, T^ny)^{2p}$$

and hence,

$$\frac{d(T^{n+1}x, T^{n+1}y)^p}{d(T^n x, T^n y)^p} \leq \frac{d(T^n x, T^n y)^p}{d(T^{n-1}x, T^{n-1}y)^p},$$

so the sequence (2.3) is monotonically decreasing. To calculate its limit in view of part (2) of Lemma 2.1, divide (2.4) by $d(T^{n-1}x, T^{n-1}y)^p$ to get

$$1 - 2 \frac{d(T^n x, T^n y)^p}{d(T^{n-1}x, T^{n-1}y)^p} + \frac{d(T^{n+1}x, T^{n+1}y)^p}{d(T^n x, T^n y)^p} \frac{d(T^n x, T^n y)^p}{d(T^{n-1}x, T^{n-1}y)^p} \leq 0.$$

and let n tend to infinity we obtain that

$$\frac{d(T^n x, T^n y)^p}{d(T^{n-1}x, T^{n-1}y)^p} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

◻

The following theorem gives a sufficient condition for (m, p) -expansive map to be $(m-1, p)$ -expansive map for $m \geq 3$.

Theorem 2.1. Let $T : X \longrightarrow X$ be an (m, p) -expansive map for $m \geq 3$. If T is $(2, p)$ -expansive, then T is $(m-1, p)$ -expansive.

Proof. The conditions $d(x, y)^p - d(Tx, Ty)^p \leq 0$ and

$$d(x, y)^p - 2d(Tx, Ty)^p + d(T^2x, T^2y)^p \leq 0$$

guarantee that the sequence $\left(d(T^{n+1}x, T^{n+1}y)^p - d(T^n x, T^n y)^p \right)_{n \geq 0}$ is monotonically non-increasing and bounded, so that it converges. Thus there exists a constant C such that

$$d(T^{n+1}x, T^{n+1}y)^p - d(T^n x, T^n y)^p \longrightarrow C \quad \text{as } n \longrightarrow \infty.$$

Since $\Theta_m^{(p)}(d, T; x, y) \leq 0$ with $m \geq 2$. By Proposition 2.1

$$\Theta_m^{(p)}(d, T; x, y) = \Theta_{m-1}^{(p)}(d, T; x, y) - \Theta_{m-1}^{(p)}(d, T; Tx, Ty),$$

we have that

$$\Theta_{m-1}^{(p)}(d, T; x, y) \leq \Theta_{m-1}^{(p)}(d, T; Tx, Ty).$$

An induction argument shows that

$$\Theta_{m-1}^{(p)}(d, T; x, y) \leq \Theta_{m-1}^{(p)}(d, T; T^n x, T^n y), \quad n \geq 1.$$

Thus, it suffices to show that

$$\Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Note that

$$\Theta_{m-1}^{(p)}(d, T; x, y) = \Theta_{m-2}^{(p)}(d, T; x, y) - \Theta_{m-2}^{(p)}(d, T; Tx, Ty),$$

so that

$$\begin{aligned} & \Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) \\ &= \sum_{0 \leq j \leq m-2} (-1)^j \binom{m-2}{j} [d(T^{n+j} x, T^{n+j} y)^p - d(T^{n+1+j} x, T^{n+1+j} y)^p]. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) \longrightarrow \sum_{0 \leq j \leq m-2} (-1)^j \binom{m-2}{j} C = 0.$$

This completes the proof. \square

Example 2.3. Let us consider again Example 2.2. $T : \mathbb{R} \longrightarrow \mathbb{R}$, $Tx = 1 + 2x$. This example shows that T is $(5, p)$ -expansive, but not $(4, p)$ -expansive, so the assumption for T to be $(2, p)$ -expansive in Theorem 2.1 below is necessary.

The following criterion for (m, p) -hyperexpansivity follows from Theorem 2.1

Corollary 2.1. Let T be (m, p) -expansive and $(2, p)$ -expansive mapping. Then T is (m, p) -hyperexpansive.

Proposition 2.3. Let $T : X \longrightarrow X$ be an $(2, p)$ -expansive map and assume that T is a (m, p) -isometric for some $m \geq 2$. Then T is a $(2, p)$ -isometric.

Proof. Assume that $\Theta_m^{(p)}(d, T; x, y) = 0$ for all $x, y \in X$. Since

$$\Theta_m^{(p)}(d, T; x, y) = \Theta_{m-1}^{(p)}(d, T; x, y) - \Theta_{m-1}^{(p)}(d, T; Tx, Ty)$$

we deduce that

$$\Theta_{m-1}^{(p)}(d, T; x, y) = \Theta_{m-1}^{(p)}(d, T; Tx, Ty) = \Theta_{m-1}^{(p)}(d, T; T^n x, T^n y);$$

for $n = 1, 2, \dots$. In the same way as in the proof of Theorem 2.1 we obtain that $\Theta_{m-1}^{(p)}(d, T; x, y) = 0$. Applying the corresponding results of the $(m-1, p)$ -isometric, we have that $\Theta_{m-2}^{(p)}(d, T; x, y) = 0$. Continue the above process to get $\Theta_2^{(p)}(d, T; x, y) = 0$ and so T is $(2, p)$ -isometric. \square

In the following lemma we generalize Lemma 1.3 in [14] and Proposition 2.11 in [13].

Lemma 2.3. Let $T : X \longrightarrow X$ be a contractive mapping. If T is an (m, p) -isometry then T is an $(m - 1, p)$ -isometry.

Proof. Since T is contractive, we have the following inequality

$d(T^{n+1}x, T^{n+1}y)^p \leq d(T^n x, T^n y)^p$ for all $x, y \in X$ and $n \in \mathbb{N}_0$. This means that $(d(T^n x, T^n y)^p)_{n \in \mathbb{N}_0}$ is a decreasing sequence, so convergent. Using the fact that T is an (m, p) -isometry and together (2.1), we obtain

$$\Theta_{m-1}^{(p)}(d, T; x, y) = \Theta_{m-1}^{(p)}(d, T; Tx, Ty) = \dots = \Theta_{m-1}^{(p)}(d, T; T^n x, T^n y).$$

Note that

$$\Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) = \Theta_{m-2}^{(p)}(d, T; T^n x, T^n y) - \Theta_{m-2}^{(p)}(d, T; T^{n+1}x, T^{n+1}y),$$

so that

$$\begin{aligned} & \Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) \\ &= \sum_{j=0}^{m-2} (-1)^j \binom{m-2}{j} [d(T^{n+j}x, T^{n+j}y)^p - d(T^{n+1+j}x, T^{n+1+j}y)^p]. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\Theta_{m-1}^{(p)}(d, T; T^n x, T^n y) \longrightarrow 0.$$

Thus, $\Theta_{m-1}^{(p)}(d, T; x, y) = 0$ and hence, T is an $(m - 1, p)$ -isometry. \square

As a consequence of the lemma, we have the following proposition.

Proposition 2.4. If T is a contractive mapping on X , then T is an (m, p) -isometry if and only if T is an isometry.

Proposition 2.5. Let $T : X \longrightarrow X$ be a map for which T^2 is isometric, then the following properties hold

- (i) T is (m, p) -expansive map if and only if T is expansive.
- (ii) T is (m, p) -contractive if and only if T is contractive.

Proof. (i) If we assume that m is odd integer i.e., $m = 2q + 1$ we have by the assumption that

$$\begin{aligned} & \Theta_{2q+1}^{(p)}(d, T; x, y) \\ &= \sum_{0 \leq j \leq q} \left[\binom{2q+1}{2j} d(T^{2j}x, T^{2j}y)^p - \binom{2q+1}{2j+1} d(T^{2j+1}x, T^{2j+1}y)^p \right] \\ &= \sum_{0 \leq j \leq q} \left[\binom{2q+1}{2j} d(x, y)^p - \binom{2q+1}{2j+1} d(Tx, Ty)^p \right] \end{aligned}$$

Since $\sum_{0 \leq k \leq 2q+1} (-1)^k \binom{2q+1}{k} = 0$, it follows that

$$\sum_{0 \leq j \leq q} \binom{2q+1}{2j+1} = \sum_{0 \leq j \leq q} \binom{2q+1}{2j}$$

and we deduce that

$$\Theta_{2q+1}^{(p)}(d, T; x, y) = \sum_{0 \leq j \leq q} \binom{2q+1}{2j} (d(x, y)^p - d(Tx, Ty)^p).$$

Similarly if m is even integer i.e., $m = 2q$ we have

$$\begin{aligned} & \Theta_{2q}^{(p)}(d, T; x, y) \\ &= \sum_{0 \leq j \leq q} \binom{2q}{2j} d(T^{2j}x, T^{2j}y)^p - \sum_{j=1}^q \binom{2q}{2j-1} d(T^{2j-1}x, T^{2j-1}y)^p \\ &= \sum_{0 \leq j \leq q} \binom{2q}{2j} d(x, y)^p - \sum_{1 \leq j \leq q} \binom{2q}{2j-1} d(Tx, Ty)^p \end{aligned}$$

Since $\sum_{0 \leq k \leq 2q} (-1)^k \binom{2q}{k} = 0$, we have that $\sum_{1 \leq j \leq q} \binom{2q}{2j-1} = \sum_{0 \leq j \leq q} \binom{2q}{2j}$ and hence

$$\Theta_{2q}^{(p)}(d, T; x, y) = \sum_{0 \leq j \leq q} \binom{2q}{2j} (d(x, y)^p - d(Tx, Ty)^p).$$

Therefore, we conclude that (i) and (ii) hold and this establishes the proposition.

QED

Proposition 2.6. If $T : X \longrightarrow X$ be an map satisfies $T^2 = T$, then the following properties hold

- (i) T is (m, p) -expansive if and only if T is expansive.
- (ii) T is (m, p) -contractive if and only if T is contractive.

Proof. By the assumption on T , we have that

$$\Theta_m^{(p)}(d, T; x, y) = d(x, y)^p - d(Tx, Ty)^p, \quad \forall x, y \in X.$$

It is clear from the foregoing that a sufficient condition for T to be (m, p) -expansive (resp. (m, p) -contractive) is that T is expansive (resp. contractive). \boxed{QED}

Proposition 2.7. ([13]) If T is a bijective (m, p) -isometry, then T^{-1} is also an (m, p) -isometry.

We have the following result about bijective (m, p) -expansive and contractive maps.

Proposition 2.8. Let $T : X \longrightarrow X$ be an bijective map, we have the following properties

- (1) If T is (m, p) -expansive, then
 - (i) for m even, T^{-1} is (m, p) -expansive.
 - (ii) for m odd, T^{-1} is (m, p) -contractive.
- (2) If T is (m, p) -contractive, then
 - (i) for m even, T^{-1} is (m, p) -contractive.
 - (ii) for m odd, T^{-1} is (m, p) -expansive.

Proof. (1) Assume that $\Theta_m^{(p)}(d, T; x, y) \leq 0 \quad \forall x, y \in X$ and for positive integer m . By a computation stemming essentially from the formula

$$\binom{m}{j} = \binom{m}{m-j}; \text{ for } j = 0, 1, \dots, m,$$

we deduce that

$$\Theta_m^{(p)}(d, T^{-1}; x, y) = (-1)^m \Theta_m^{(p)}(d, T; T^{-m}x, T^{-m}y).$$

It follows that $\Theta_m^{(p)}(d, T^{-1}; x, y) \leq 0$ for even integer m i.e., T^{-1} is (m, p) -expansive, and $\Theta_m^{(p)}(d, T^{-1}; x, y) \geq 0$ for odd integer m i.e., T^{-1} is (m, p) -contractive.

- (2) The proof is similar. \boxed{QED}

Proposition 2.9. Let $T : X \longrightarrow X$ be an bijective $(2, p)$ -expansive map, then T is $(1, p)$ -isometric.

Proof. Since T is $(2, p)$ -expansive, we have by Lemma 2.1 that $d(Tx, Ty)^p \geq d(x, y)^p$. This means that T is $(1, p)$ -expansive. Moreover if T is bijective $(2, p)$ -expansive, then T^{-1} is $(2, p)$ -expansive, hence $d(T^{-1}u, T^{-1}v)^p \geq d(u, v)^p$ for all $u, v \in X$. Letting $u = Tx$ and $v = Ty$, this implies

$$d(Tx, Ty)^p = d(x, y)^p$$

for all $x, y \in X$. This means that T is $(1, p)$ -isometric. \square

Theorem 2.2. Let $T, S : X \longrightarrow X$ two maps such that $ST = I_X$ (the identity mapping). Assume that there exists an integer $m \geq 1$ such that $\Theta_m^{(p)}(d, S; x, y) \leq 0$ for all $x, y \in \mathcal{R}(T^m)$, the following statements hold.

- (i) If m is even, then T is (m, p) -expansive map.
- (ii) If m is odd, then T is (m, p) -contractive map.

Proof. Since $\Theta_m^{(p)}(d, S; T^m u, T^m v) \leq 0$ for all $u, v \in X$, we have that

$$\begin{aligned} 0 &\geq \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(S^k T^m u, S^k T^m v)^p \\ &\geq \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(T^{m-k} u, T^{m-k} v)^p \\ &\geq (-1)^m \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} d(T^k u, T^k v)^p \end{aligned}$$

\square

Note that every power of k -expansive (resp. (A, m) -expansive) operators on a Hilbert space is k -expansive (resp. (A, m) -expansive). See ([18], Theorem 2.3) and ([20], Proposition 3.9).

In the following theorem we investigate the powers of $(2, p)$ -expansive maps as well as $(2, p)$ -expansive maps by using Lemma 2.2.

Theorem 2.3. Let $T : X \longrightarrow X$ be an $(2, p)$ -expansive map. Then for any positive integer n , T^n is $(2, p)$ -expansive map.

Proof. We will induct on n , the result obviously holds for $n = 1$. Suppose then the assertion holds for $n \geq 2$, i.e

$$d(T^{2n}x, T^{2n}y)^p - 2d(T^n x, T^n y)^p + d(x, y)^p \leq 0, \forall x, y \in X.$$

Then

$$\begin{aligned} & d(T^{2n+2}x, T^{2n+2}y)^p - 2d(T^{n+1}x, T^{n+1}y)^p + d(x, y)^p \\ = & d(T^2 T^{2n}x, T^2 T^{2n}y)^p - 2d(T^{n+1}x, T^{n+1}y)^p + d(x, y)^p \\ \leq & 2d(T^{2n+1}x, T^{2n+1}y)^p - d(T^{2n}x, T^{2n}y)^p \\ & - 2d(T^{n+1}x, T^{n+1}y)^p + d(x, y)^p \\ \leq & 2(2d(T^{n+1}x, T^{n+1}y)^p - d(Tx, Ty)^p) - d(T^{2n}x, T^{2n}y)^p \\ & - 2d(T^{n+1}x, T^{n+1}y)^p + d(x, y)^p \\ \leq & 2d(T^{n+1}x, T^{n+1}y)^p - d(T^{2n}x, T^{2n}y)^p \\ & - 2d(Tx, Ty)^p + d(x, y)^p \\ \leq & 2d(T^{n+1}x, T^{n+1}y)^p - (2d(T^n x, T^n y)^p - d(x, y)^p) \\ & - 2d(Tx, Ty)^p + d(x, y)^p \\ \leq & 2d(T^{n+1}x, T^{n+1}y)^2 - 2d(T^n x, T^n y)^p - 2d(Tx, Ty)^p + 2d(x, y)^p \\ \leq & 2(d(Tx, Ty)^p - d(x, y)^p) - 2d(Tx, Ty)^p + 2d(x, y)^p \quad (\text{by Lemma 2.2}). \\ \leq & 0. \end{aligned}$$

Thus means that T^n is $(2, p)$ -expansive map. \square QED

In the following theorem we investigate the powers of completely p -hyperexpansive mapping as well as completely p -hyperexpansive mapping.

According to [5, Remark 1.] for every completely p -hyperexpansive map, the condition that $n \longmapsto d(T^n x, T^n y)^p$ be completely alternating on \mathbb{N} implies the representation, for every $x, y \in X$,

$$d(T^n x, T^n y)^p = d(x, y)^p + n\mu_{x,y}(\{1\}) + \int_{[0,1)} (1-t^n) \frac{d\mu_{x,y}(t)}{1-t}, \quad (2.5)$$

where $\mu_{x,y}$ is a positive regular Borel measure on $[0; 1]$ (for more details see [5]).

Theorem 2.4. Any positive integral power of a completely p -hyperexpansive mapping is completely p -hyperexpansive.

Proof. Let T be a completely p -hyperexpansive map and let $k \geq 1$. In view of (2.5) we have that

$$\begin{aligned} d((T^k)^n x, (T^k)^n y)^p &= d(T^{nk} x, T^{nk} y)^p \\ &= d(x, y)^p + nk\mu_{x,y}(\{1\}) + \int_{[0,1)} (1-t^{nk}) \frac{d\mu_{x,y}(t)}{1-t} \\ &= d(x, y)^p + n(k\mu_{x,y}(\{1\})) + \int_{[0,1)} (1-s^n) \frac{d\mu'_{x,y}(s)}{1-s^{\frac{1}{k}}}. \end{aligned}$$

Therefore the map $n \mapsto d(T^{nk} x, T^{nk} y)^p$ is completely alternating and so that T^k is completely p -hyperexpansive. \square

The next proposition describes the intersection of the class of completely p -hyperexpansive maps with the class of (m, p) -isometries.

Proposition 2.10. Let T be a mapping on metric space X into itself. If T is completely p -hyperexpansive as well as (m, p) -isometric ($m \geq 2$), then T is a $(2, p)$ -isometric.

Proof. First, if T is isometric, then T is a $(2, p)$ -isometric. Assume that T is a (m, p) -isometric with $m \geq 2$, then we have that $\Theta_m^{(p)}(d, T; x, y) = 0$ and from (2.5) it follows that

$$\begin{aligned} 0 &= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} k\mu_{x,y}(\{1\}) \\ &\quad + \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \int_{[0,1)} (1-t^k) \frac{d\mu_{x,y}(t)}{1-t} \\ &= - \int_{[0,1)} (1-t)^{m-1} d\mu_{x,y}(t) \end{aligned}$$

Now $\int_{[0,1)} (1-t)^{m-1} d\mu_{x,y}(t) = 0$ gives that

$$d(T^k x, T^k y)^p = d(x, y)^p + k\mu_{x,y}(\{1\}) \quad \text{for all } k$$

and therefore

$$\Theta_2^{(p)}(d, T; x, y) = 0.$$

\square

Example 2.4. Consider the map $T : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d)$ defined by

$$Tx = \begin{cases} 2x - 1, & \text{for } x \leq 0 \\ 2x + 1 & \text{for } x > 0. \end{cases}$$

It is easy to verify that T is a $(1, p)$ -expansive, but T is neither continuous nor linear.

3 (m, p) -hyperexpansive maps in seminormed space

Let X be a linear vector space and considering a seminorm s on X , we may define the quantity

$$\Theta_m^{(p)}(s, T; x) := \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} s(T^k x)^p$$

for all $x \in X$, and introducing a concept similar to that of (m, p) -expansive maps.

Definition 3.1. Let $T : X \longrightarrow X$ be a map, $m \in \mathbb{N}$ and $p > 0$. We say that

- (1) T is $s(m, p)$ -isometric if $\Theta_m^{(p)}(s, T; x) = 0$ for all $x \in X$.
- (2) T is $s(m, p)$ -expansive if $\Theta_m^{(p)}(s, T; x) \leq 0 \quad \forall x \in X$
- (3) T is $s(m, p)$ -hyperexpansive if $\Theta_k^{(p)}(s, T; x) \leq 0$ for $k = 1, \dots, m$ and $x \in X$.
- (4) T is completely s -hyperexpansive if T is $s(k, p)$ -expansive for all $k \in \mathbb{N}$.
- (5) T is $s(m, p)$ -contractive if $\Theta_m^{(p)}(s, T; x) \geq 0 \quad \forall x \in X$.
- (6) T is $s(m, p)$ -hypercontractive if $\Theta_k^{(p)}(s, T; x) \geq 0$ for $k = 1, 2, \dots, m$ and $x \in X$.
- (7) T is completely s -hypercontractive if T is $s(k, p)$ -contractive for all $k \in \mathbb{N}$.

For any $p > 0$, $s(1, p)$ -expansive coincide with s -expansive; that is, maps T satisfying $s(Tx) \geq s(x)$, for all $x \in X$. Every s -isometry is an $s(m, p)$ -isometry for all $m \geq 1$ and $p > 0$. $s(m, p)$ -isometries maps are special cases of the class of $s(m, p)$ -expansive maps.

If X is a normed space with norm $\|\cdot\|$ and $T : X \longrightarrow X$ we have that

$$\Theta_m^{(p)}(\|\cdot\|, T; x) = \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|T^k x\|^p.$$

Clearly m -hyperexpansivity on Hilbert spaces agree with $(m, 2)$ -hyperexpansivity.

Example 3.1. Let \mathbb{D} denote the open unit disk in the complex plane. An analytic function f on \mathbb{D} is said to be Bloch if

$$s(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The mapping $f \mapsto s(f)$ is a semi-norm on the space \mathcal{B} of Bloch functions, called the Bloch space. See [24] for some additional details. Consider for $\lambda \in \mathbb{C}$ the map $T_\lambda : \mathcal{B} \rightarrow \mathcal{B} : T_\lambda(f) = \lambda f$.

A simple computation shows that

$$\begin{aligned} & \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} s(T_\lambda^k f) \\ &= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} |\lambda|^k \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \\ &= (1 - |\lambda|)^m \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \end{aligned}$$

and it follows that

$$\left\{ \begin{array}{l} \Theta_m^{(p)}(s, T_\lambda, f) \leq 0, \text{ if } |\lambda| > 1, \text{ for odd } m. \\ \Theta_m^{(p)}(s, T_\lambda, f) = 0 \text{ if } |\lambda| = 1 \text{ and for all } m \\ \Theta_m^{(p)}(s, T_\lambda, f) > 0 \text{ if } |\lambda| < 1 \text{ and for all } m. \end{array} \right.$$

Setting

$$\beta_k^{(p)}(s, T; x) := \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} s(T^j x)^p, \quad \forall x \in X. \quad (3.1)$$

In the following theorem, we generalized the identities (1.7) and (1.10) to seminormed space. We omit the proof which is very similar to [13, Theorem 2.5] and [8, Proposition 2.1].

Theorem 3.1. Let (X, s) be seminormed space and $T : X \rightarrow X$ be a map, we have that

$$(i) \quad s(T^n x)^p = \sum_{0 \leq j \leq n} n^{(j)} \beta_j^{(p)}(s, T; x); \quad \forall n \in \mathbb{N}.$$

(ii) T is an $s(m, p)$ -isometry if and only if

$$s(T^n x)^p = \sum_{0 \leq j \leq m-1} n^{(j)} \beta_j^{(p)}(s, T; x); \quad \forall n \in \mathbb{N}.$$

Proposition 3.1. Let (X, s) be a seminormed space and $T : X \longrightarrow X$ be a map. The following are true

(i) T is $s(m, p)$ -expansive if and only if, λT is $s(m, p)$ -expansive for all $\lambda \in \mathbb{C}$: $|\lambda| = 1$,

(ii) If T is $s(2, p)$ -expansive, then

(1) λT is $s(2, p)$ -expansive for $|\lambda| < 1$, if λT^2 is s -expansive.

(2) λT is $s(2, p)$ -expansive for $|\lambda| > 1$, if λT^2 is s -contractive.

Proof. (i) Note that, for all $p > 0$, $\lambda \in \mathbb{C}$, and all $x \in X$, we have

$$\Theta_m^{(p)}(s, T x) = \Theta_m^{(p)}(s, \lambda T; x), \quad |\lambda| = 1.$$

(ii) If T is $s(2, p)$ -expansive, then

$$-2|\lambda|^p s(Tx)^p \leq |\lambda|^p [-s(T^2x)^p - s(x)^p] \quad \text{for every } \lambda \in \mathbb{C}.$$

So we have for every $\lambda \in \mathbb{C}$

$$|\lambda|^{2p} s(T^2x)^p - 2|\lambda|^p s(Tx)^p + s(x)^p \leq (|\lambda|^p - 1)(|\lambda|^p s(T^2x)^p - s(x)^p)$$

This finishes the proof. \square

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